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BEHAVIOUR OF VISCOELASTIC-VISCOPLASTIC SPHERES AND CYLINDERS—PARTLY PLASTIC VESSEL WALLS

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Abstract—The material model consists of a viscoelastic Burgers element and an additional viscoplastic Bingham element when the effective stress exceeds the yield stress. For partly plastic vessel walls, expressions are derived for the stress and strain state in pressurised or relaxation loaded thick-walled cylinders in plane strain and spheres. For the spherical problem, the material compressibility is accounted for. The influence of the different material parameters on the behaviour of the vessels is evaluated. It is shown that the magnitude of the Maxwell viscosity is of major importance for the long-term behaviour of thick-walled partly plastic vessels.

INTRODUCTION

The quasi-static time-dependent behaviour of thick-walled spheres and cylinders is of great importance in many areas ranging from mechanical design situations to rock mechanical problems. Typically, the creep behaviour of materials changes from a linear dependence for small stresses to a nonlinear stress dependence for higher loadings. A combined viscoelastic-viscoplastic model can simulate such a material behaviour. Below a certain yield stress, a linear viscoelastic behaviour occurs, whereas additional viscoplastic response occurs if this yield stress is exceeded. However, only a few simplified analytical solutions for the behaviour of thick-walled vessels have been established for such a material behaviour, and numerical solutions are most often resorted to.

It will turn out that there is a fundamental difference in the behaviour of the vessel depending on whether it is partly or fully plastic. In the first case, we have, in general, a moving plastic boundary, which presents a much more difficult problem than when the vessel is fully plastic. Partly plastic vessels will be treated in the present article, whereas fully plastic vessels are considered in [1].

For a partly plastic pressurised sphere, Madejski[2] established a solution for the stress field, adopting a simplified elastic-viscoplastic behaviour. However, it will be shown that the solution of Madejski[2], often referred to, is erroneous. Very recently, Berest and Nguyen[3] presented the correct solution to this elastic-viscoplastic spherical problem, accounting even for a linear hardening of the viscoplastic behaviour. For the same elastic-viscoplastic material, Nonaka[4] treated the relaxation behaviour of infinite spheres and cylinders, but the assumptions of incompressibility and a fixed prescribed displacement indicate that the plastic radius becomes constant with time, which simplifies the analysis significantly.

The solutions derived in this study for partly plastic vessels represent significant generalisations and extensions. First, incompressible cylinders in plane strain and spheres are treated in a unified fashion. Second, pressurised as well as relaxation loaded vessels are treated. Even for these widely different cases, it is possible to follow a unified exposition fairly extensively. In addition, we apply a quite general constitutive model consisting of a viscoelastic Burgers model below the yield stress as well as a viscoplastic Bingham model above this limit. For the spherical problem, the material compressibility is accounted for.

For this broad spectrum of problems, we shall present solutions of the stress and strain fields, and a detailed discussion will be given with emphasis on the principal aspects of the vessel behaviour and the influence of the different material parameters.

CONSTITUTIVE EQUATIONS

We shall now set up the constitutive equations for a material having a creep-sensitive deviatoric response while its volumetric response is purely elastic. Below a certain yield stress, we assume a viscoelastic response corresponding to a Burgers material, i.e. a Maxwell and a Kelvin element in series. Above the yield stress, we assume an additional viscoplastic response corresponding to a Bingham element. The material response is symbolised in Fig. 1.

It appears that the material model reflects many creep characteristics that can be observed for metals, soils and rocks. Owing to the friction element, the creep behaviour depends nonlinearly on the stresses, and whereas the Maxwell and the Bingham elements exhibit secondary, irreversible creep, the Kelvin element exhibits primary, reversible creep. The presence of the Kelvin element is therefore important for many applications. However, its inclusion complicates the calculations considerably as it increases the order of the involved differential equations from one to two.

The deviatoric stress and strain tensor are defined by

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk}, \qquad e_{ij} = \epsilon_{ij} - \frac{1}{3}\delta_{ij}\epsilon_{kk}, \tag{1, 2}$$

where σ_{ij} and ϵ_{ij} denote the stress and strain tensor, respectively, and where usual tensor notation is applied. Tension and elongation are considered positive.

For the Maxwell, Kelvin and Bingham elements, all quantities are labelled with the index M, K and B, respectively. Moreover, the constants G and η denote, in general, a shear modulus and a viscosity coefficient, respectively.

The constitutive equation for the Maxwell element is

$$e_{ij,M} = \frac{s_{ij}}{2G_M} + \frac{1}{2\eta_M}\int s_{ij}\,\mathrm{d}t. \tag{3}$$

The constitutive equation for the Kelvin element is

$$s_{ij} = 2\eta_K\,\dot{e}_{ij,K} + 2G_K\,e_{ij,K}, \tag{4}$$

where a dot denotes the time derivative. For stresses below the yield stress, the Bingham element is rigid; otherwise, the constitutive equation for the Bingham element is

$$e_{ij,B} = \frac{1}{2\eta_B}\int\left(1 - \frac{\sigma_y}{\sigma_e}\right)s_{ij}\,\mathrm{d}t, \quad \text{for} \quad \sigma_e \geqslant \sigma_y, \tag{5}$$

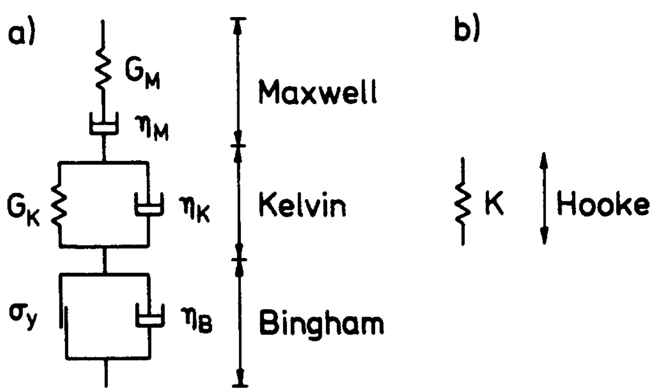

Fig. 1. Material model: (a) deviatoric response, (b) volumetric response.

where σ_y is the yield stress and σ_e is the effective stress of von Mises defined by

$$\sigma_e = \left(\frac{3}{2} s_{ij} s_{ij}\right)^{1/2}. \tag{6}$$

Assuming small strains, the total deviatoric strain tensor e_{ij} consists of the contributions from the Maxwell, Kelvin and Bingham elements, i.e.

$$e_{ij,K} = e_{ij} - e_{ij,M} - e_{ij,B}. \tag{7}$$

Inserting (7) in (4) and using (3) and (5) yields, after differentiation,

$$\frac{\eta_K}{G_M}\ddot{s}_{ij} + \left[1 + \frac{\eta_K}{\eta_M} + \frac{G_K}{G_M} + \frac{\eta_K}{\eta_B}\left(1 - \frac{\sigma_y}{\sigma_e}\right)\right]\dot{s}_{ij}$$
$$+ \left[\frac{\eta_K\,\sigma_y\dot{\sigma}_e}{\eta_B\,\sigma_e^2} + \frac{G_K}{\eta_M} + \frac{G_K}{\eta_B}\left(1 - \frac{\sigma_y}{\sigma_e}\right)\right]s_{ij} = 2\eta_K\,\ddot{e}_{ij} + 2G_K\,\dot{e}_{ij}. \tag{8}$$

This second-order differential equation governs the deviatoric response of the material. Its solution requires the specification of two initial conditions. Here we assume that the load is applied suddenly; i.e. an elastic response exists at $t = 0^+$. As a consequence, all the initial displacement occurs in the Maxwell spring; i.e. the first initial condition becomes

$$s_{ij,0} = 2G_M\,e_{ij,0}, \tag{9}$$

where, for instance, $e_{ij,0}$ denotes the value of e_{ij} at $t = 0^+$. Initially, the Kelvin element is completely undeformed, i.e. $e_{ij,K} = 0$. Therefore, inserting (7) in (4) and using (3) and (5) yields the second initial condition:

$$\dot{s}_{ij,0} = 2G_M\,\dot{e}_{ij,0} - G_M\left[\frac{1}{\eta_M} + \frac{1}{\eta_K} + \frac{1}{\eta_B}\left(1 - \frac{\sigma_y}{\sigma_{e,0}}\right)\right]s_{ij,0}. \tag{10}$$

Clearly, by letting $\eta_B \to \infty$, (8)–(10) apply also when the stresses are below the yield stress.

Contrary to the viscous deviatoric response as defined above, the volumetric behaviour is assumed to be purely elastic, i.e.

$$\epsilon_{ii} = \frac{\sigma_{ii}}{3K}, \tag{11}$$

where the constant K is the bulk modulus.

The constitutive equations derived above apply generally, but they shall later be used for spherical and cylindrical vessels. Before this, we shall derive some general expressions for such structures, and in particular a unified formulation of incompressible cylinders in plane strain and spheres will be established.

SPHERICAL PROBLEM

For spheres, only two directions are of interest. It follows from (6) that

$$\sigma_e = T(\sigma_\theta - \sigma_r) \tag{12}$$

in familiar notation. Depending on the problem, we choose $T = 1$ or $T = -1$ so as to ensure that the effective stress is positive. We also have

$$s_\theta = \frac{\sigma_e}{3T}. \tag{13}$$

The compatibility equation, which applies to spherical as well as cylindrical problems, reads

$$r\frac{\partial \epsilon_\theta}{\partial r} + \epsilon_\theta - \epsilon_r = 0. \tag{14}$$

From spherical symmetry, it follows that

$$\epsilon_r = \epsilon_{ii} - 2\epsilon_\theta, \qquad \sigma_{ii} = 2T\sigma_e + 3\sigma_r. \tag{15, 16}$$

Inserting (15) in (14) and making use of (11) and (16) yields

$$r\frac{\partial \epsilon_\theta}{\partial r} + 3\epsilon_\theta = \frac{1}{3K}(2T\sigma_e + 3\sigma_r). \tag{17}$$

The equilibrium equation for spherical problems can be written

$$r\frac{\partial \sigma_r}{\partial r} = 2T\sigma_e. \tag{18}$$

The only variables are radius and time. Therefore, inserting (18) in (17) and integrating gives

$$\epsilon_\theta = \frac{f(t)}{r^3} + \frac{\sigma_r}{3K}, \tag{19}$$

where $f(t)$ is an arbitrary time-dependent function. The only assumption related to (19) is that of Hooke's law for volumetric response. Apart from that, (19) applies generally; i.e. $f(t)$ is the same universal function all through the vessel wall.

As $e_\theta = \epsilon_\theta - \sigma_{ii}/9K$, the use of (16) and (19) results in

$$e_\theta = \frac{f(t)}{r^3} - \frac{2T}{9K}\sigma_e. \tag{20}$$

By definition, $\epsilon_r = -2\epsilon_\theta + \sigma_{ii}/3K$. On eliminating ϵ_θ and σ_{ii} by means of (19) and (16), respectively, we find

$$\epsilon_r = -2\frac{f(t)}{r^3} + \frac{1}{3K}(2T\sigma_e + \sigma_r). \tag{21}$$

CYLINDRICAL PROBLEM

To attain closed-form solutions for the cylindrical problem, it becomes necessary to assume incompressibility; this implies that Poisson's ratio equals $\frac{1}{2}$, whereby $\epsilon_{ij} = e_{ij}$. Moreover, only plane strain is considered, i.e. $\epsilon_z = 0$. From these assumptions, there follows

$$\epsilon_r = -\epsilon_\theta. \tag{22}$$

The only variables are time and radius. Therefore, using (22) in the compatibility equation, as given by (14), and integrating result in

$$e_\theta = \epsilon_\theta = \frac{f(t)}{r^2}, \tag{23}$$

where $f(t)$ is an arbitrary time-dependent function. The only assumptions related to (23) are those of incompressibility and plane strain. Apart from those, (23) applies generally; i.e. $f(t)$ is the same universal function all through the vessel wall.

Considering the z-direction and noting that $e_z = 0$, (9) shows that the first initial condition is $s_{z,0} = 0$, and (10) shows that the second initial condition is $\dot{s}_{z,0} = 0$. Moreover, in the z-direction, constitutive equation (8) reduces to a homogeneous differential equation, where the trivial solution $s_z = 0$ is the true solution, as it satisfies the above-mentioned initial conditions. Consequently,

$$\sigma_z = \frac{1}{2}(\sigma_r + \sigma_\theta) \tag{24}$$

and the effective stress as given by (6) becomes

$$\sigma_e = T\frac{\sqrt{3}}{2}(\sigma_\theta - \sigma_r). \tag{25}$$

Depending on the problem, we choose again $T = 1$ or $T = -1$ to ensure that the effective stress is positive. From (24) and (25), there follows

$$s_\theta = \frac{\sigma_e}{T\sqrt{3}}. \tag{26}$$

Using (25), the equilibrium equation for cylindrical problems can be written as

$$r\frac{\partial \sigma_r}{\partial r} = \frac{2T}{\sqrt{3}}\sigma_e. \tag{27}$$

UNIFIED FORMULATION FOR SPHERES AND CYLINDERS

To facilitate the exposition, a unified treatment of the spherical and cylindrical problems is clearly preferable. This can be attained by writing the equilibrium equation as

$$\frac{\partial \sigma_r}{\partial r} = \lambda\frac{\sigma_e}{r}. \tag{28}$$

A comparison with (18) and (27) shows that

$$\lambda = \begin{cases} 2T, & \text{for spherical problems} \\ \dfrac{2T}{\sqrt{3}}, & \text{for cylindrical problems.} \end{cases} \tag{29}$$

Similarly, (13) and (26) show that

$$s_\theta = \frac{2\sigma_e}{3\lambda}. \tag{30}$$

Finally, (20) and (23), which are derived from the compatibility equation, show that we can write

$$e_\theta = \frac{f(t)}{r^\alpha} - M\,\sigma_e, \tag{31}$$

where

$$\alpha = \begin{cases} 3, & \text{for spherical problems} \\ 2, & \text{for cylindrical problems} \end{cases} \tag{32}$$

and where

$$M = \begin{cases} \dfrac{2T}{9K}, & \text{for spherical problems} \\ 0, & \text{for cylindrical problems.} \end{cases} \tag{33}$$

In general, incompressibility, i.e. $K \to \infty$, implies that $M = 0$. Note that $f(t)$ is the same universal function all through the vessel wall.

Thus, we have now formulated a problem where the equilibrium equation is given by (28) and the compatibility equation is stated as (31). The constitutive equation for the deviatoric response and the corresponding initial conditions are given by (8)–(10). The volumetric response is defined by (11). What remains is to specify the boundary conditions. Denoting the inner radius by r_1 and the outer one by r_2, the boundary conditions are

$$r = r_2, \qquad \sigma_r = -p_2(t) \tag{34}$$

$$r = r_1, \quad \begin{cases} \sigma_r = -p_1(t), & \text{stress boundary problem} \\ u = u_1, & \text{displacement boundary problem.} \end{cases} \tag{35}$$

That is, we will consider pressurised vessels as well as relaxation of vessels. However, even for the displacement boundary problem, there exists a pressure along the inner surface, and this so-called shrink-fit pressure varies with time. Therefore, the boundary condition (35) suggests that

$$r = r_1, \qquad \sigma_r = -p_1(t) \tag{36}$$

always applies. For relaxation problems, the shrink-fit pressure is still unknown. It will appear that it can be determined using (35).

We shall also state expressions for the circumferential and radial strain. From (19) and (23), valid for the spherical and cylindrical problems, respectively, the circumferential strain, defined by $\epsilon_\theta = u/r$ where u is the radial displacement, can be written in the following unified way:

$$\epsilon_\theta = \frac{f(t)}{r^\alpha} + \frac{3M}{2T}\,\sigma_r. \tag{37}$$

Similarly, (21) and (22) indicate that

$$\epsilon_r = -(\alpha - 1)\,\frac{f(t)}{r^\alpha} + \frac{3M}{2T}\,(2T\sigma_e + \sigma_r). \tag{38}$$

Note that for incompressible materials, where $M = 0$ [see (33)], the strains depend on the radius through the factor $r^{-\alpha}$ alone, irrespective of the material model.

It should be observed that the preceding unified formulation of cylinders and spheres applies quite generally. For the spherical problem, the only assumption used is that of Hooke's law for volumetric response. For the cylindrical problem, the only assumptions are those of incompressibility, plane strain and that $s_z = 0$ applies.

PRELIMINARY EXPRESSIONS FOR THE STRESSES AND STRAINS

Having formulated the constitutive equations, the initial and boundary conditions as well as some general expressions for the vessels, we are now in a position to derive some general preliminary expressions for the effective stress and the time-dependent function $f(t)$ present, for instance, in (31). Later on, these expressions shall be simplified, but this simplification depends on whether or not all stress states in the vessel wall exceed the yield stress.

From spherical symmetry, there follows $s_r = -2s_\theta$ and $e_r = -2e_\theta$. For the cylindrical problem, we have $s_r = -s_\theta$ and $e_r = -e_\theta$. Therefore, the deviatoric constitutive equation (8) takes the same form whether the radial or the circumferential direction is considered. Taking, for convenience, the circumferential direction and using (30) and (31), constitutive equation (8) can, after some algebra, be written as

$$\ddot{\sigma}_e + A\dot{\sigma}_e + B\sigma_e = C\frac{\ddot{f}}{r^\alpha} + D\frac{\dot{f}}{r^\alpha} + F, \tag{39}$$

where

$$A = \frac{G_M}{1 + 3\lambda M\, G_M}\left(\frac{1}{\eta_M} + \frac{1}{\eta_K} + \frac{1}{\eta_B}\right) + \frac{G_K}{\eta_K} \tag{40}$$

$$B = \frac{G_M\, G_K}{(1 + 3\lambda M\, G_M)\eta_K}\left(\frac{1}{\eta_M} + \frac{1}{\eta_B}\right) \tag{41}$$

$$C = \frac{3\lambda\, G_M}{1 + 3\lambda M\, G_M} \tag{42}$$

$$D = \frac{3\lambda\, G_M\, G_K}{(1 + 3\lambda M\, G_M)\eta_K} \tag{43}$$

$$F = \frac{G_M\, G_K\, \sigma_y}{(1 + 3\lambda M\, G_M)\eta_K\, \eta_B} \tag{44}$$

and where the term λM is always nonnegative. Note that constitutive equation (39) applies always so long as the material point is in a viscoelastic-viscoplastic state.

Let us now assume that the material point initially is also in a viscoelastic-viscoplastic state. Then, the two initial conditions (9) and (10) apply, and using (30) and (31) they can be written as

$$\sigma_{e,0} = C\frac{f_0}{r^\alpha} \tag{45}$$

and

$$\dot{\sigma}_{e,0} = C\frac{\dot{f}_0}{r^\alpha} - (AC - D)\frac{f_0}{r^\alpha} + N, \tag{46}$$

where $D/C = G_K/\eta_K$ has been used and where

$$N = \frac{G_M\, \sigma_y}{(1 + 3\lambda M\, G_M)\eta_B}. \tag{47}$$

The roots of the characteristic equation belonging to (39) are

$$\left.\begin{matrix} R' \ (\leq 0) \\ R'' \ (<0) \end{matrix}\right\} = \frac{1}{2}(-A \pm \sqrt{A^2 - 4B}. \tag{48}$$

Trivial considerations show that $A^2 - 4B \geq 0$ always holds. The differential equation (39) subjected to the initial conditions (45) and (46) has been solved in Appendix A. Therefore, using (A4) we can write the solution to (39) directly as

$$\sigma_e = \Phi(t) + \frac{Cf(t) + \psi(t)}{r^\alpha}, \tag{49}$$

where the function $\Phi(t)$ is defined by

$$\Phi(t) = \frac{e^{R't} - e^{R''t}}{R' - R''}\left(N + \frac{\sigma_y R''}{1 + \eta_B/\eta_M}\right) + \frac{\sigma_y}{1 + \eta_B/\eta_M}(1 - e^{R''t}) \tag{50}$$

and where the relation $F/B = \sigma_y/(1 + \eta_B/\eta_M)$ has been used. The unknown function $\psi(t)$ present in (49) is given by (A2), i.e.

$$\psi(t) = -\frac{1}{R' - R''}\left[R''(CR'' + D)\, e^{R''t} \int_0^t f(t)\, e^{-R''t}\, dt \right.$$

$$\left. - R'(CR' + D)\, e^{R't} \int_0^t f(t)\, e^{-R't}\, dt\right]. \tag{51}$$

Note that solution (49) applies to material points that initially are in a viscoelastic-viscoplastic state and that remain in such a state. The fundamental property of (49) is that the effective stress depends on radius and time through separated functions.

If a material point is in a viscoelastic state, the constitutive equation can be obtained from (39) by letting the viscoplastic viscosity $\eta_B \to \infty$ in the expressions for the parameters. The derived parameters are labelled with subscript e; i.e. we obtain

$$A_e = \frac{G_M}{1 + 3\lambda M\, G_M}\left(\frac{1}{\eta_M} + \frac{1}{\eta_K}\right) + \frac{G_K}{\eta_K} \tag{52}$$

$$B_e = \frac{G_M\, G_K}{(1 + 3\lambda M\, G_M)\eta_K\, \eta_M} \tag{53}$$

$$C_e = C \tag{54}$$

$$D_e = D \tag{55}$$

$$F_e = 0 \tag{56}$$

$$N_e = 0. \tag{57}$$

The constitutive equation then becomes

$$\ddot{\sigma}_e + A_e\dot{\sigma}_e + B_e\sigma_e = C\frac{\ddot{f}}{r^\alpha} + D\frac{\dot{f}}{r^\alpha}, \tag{58}$$

and this expression applies always so long as the material point is in a viscoelastic state.

The roots of the characteristic equation are

$$\left.\begin{matrix} R'_e \ (\leq 0) \\ R''_e \ (<0) \end{matrix}\right\} = \frac{1}{2}(-A_e \pm \sqrt{A_e^2 - 4B_e}), \tag{59}$$

where $A_e^2 - 4B_e \geq 0$ holds again.

Assume now that the material point initially also is in a viscoelastic state. The initial conditions are then given similarly to (45) and (46) by replacing A, C, D and N with the corresponding parameters given by (52), (54), (55) and (57). Consequently, we can state the solution to (58) directly from (49), noting that $\phi(t)_e = 0$ because $\eta_B \to \infty$ and $N_e = 0$. The solution is therefore

$$\sigma_e = \frac{Cf(t) + \psi(t)_e}{r^\alpha}, \tag{60}$$

where

$$\psi(t)_e = -\frac{1}{R'_e - R''_e}\left[R''_e(CR''_e + D)\,e^{R''_e t}\int_0^t f(t)\,e^{-R''_e t}\,dt - R'_e(CR'_e + D)\,e^{R'_e t}\int_0^t f(t)\,e^{-R'_e t}\,dt\right]. \tag{61}$$

The fundamental property of (60) is that the effective stress depends on radius and time through separated functions.

Note that (49) applies to material points that always remain in a viscoelastic-viscoplastic state, whereas (60) applies to material points that always remain in a viscoelastic state.

For material points that change from a viscoelastic-viscoplastic state to a viscoelastic state or vice versa, the functions $\psi(t)$ and $\psi(t)_e$ will take different forms as they will depend on the time when the material state is changed; i.e. they will take different values for different positions. This aspect was not noticed in the often-referred-to paper by Madejski[2], and as a consequence the solution derived in [2] for a partly plastified elastic-viscoplastic sphere is erroneous.

We shall now determine a preliminary expression for the time-dependent function $f(t)$ present in the strain expressions, as, for instance, (31). For material points that always are in a viscoelastic-viscoplastic state, integration of (39) gives [see (B3)]

$$\frac{f(t)}{r^\alpha} = \frac{1}{3\lambda}\left[\frac{3\lambda}{C}\sigma_e + \frac{e^{(-G_K/\eta_K)t}}{\eta_K}\int_0^t e^{(G_K/\eta_K)t}\,dt + \left(\frac{1}{\eta_M} + \frac{1}{\eta_B}\right)\int_0^t \sigma_e\,dt - \frac{\sigma_y}{\eta_B}t\right]. \tag{62}$$

By letting $\eta_B \to \infty$, we obtain for material points that always are in a viscoelastic state

$$\frac{f(t)}{r^\alpha} = \frac{1}{3\lambda}\left[\frac{3\lambda}{C}\sigma_e + \frac{e^{(-G_K/\eta_K)t}}{\eta_K}\int_0^t \sigma_e\,e^{(G_K/\eta_K)t}\,dt + \frac{1}{\eta_M}\int_0^t \sigma_e\,dt\right]. \tag{63}$$

The discussion above indicates that there is a fundamental difference whether or not we have a moving plastic boundary, i.e. whether or not we have fully or partly plastic vessel walls. In the following exposition, we shall concentrate on partly plastic vessels, whereas fully plastic vessels are treated in [1].

A VISCOELASTIC ZONE EXISTS

Let the time-dependent plastic radius $r_y(t)$ denote the radius, which divides the vessel into a viscoelastic-viscoplastic region and a viscoelastic region. The viscoplastic region is always present adjacent to the inner boundary. In the first place, we shall derive an expression that determines the development with time of the plastic radius.

Irrespective of the previous stress history, the constitutive equation (39) applies in the whole viscoelastic-viscoplastic region and (58) applies in the whole viscoelastic region. Multiply (39) by λ/r and integrate from r_1 to r_y; similarly, multiply (58) by λ/r and integrate from r_y to r_2. Adding the resulting two expressions yields

$$\lambda \int_{r_1}^{r_y} \frac{\ddot{\sigma}_e + A\dot{\sigma}_e + B\sigma_e - F}{r}\,\mathrm{d}r + \lambda \int_{r_y}^{r_2} \frac{\ddot{\sigma}_e + A_e\dot{\sigma}_e + B_e\sigma_e}{r}\,\mathrm{d}r = \frac{\lambda}{\alpha}(C\ddot{f} + D\dot{f})\left(\frac{1}{r_1^\alpha} - \frac{1}{r_2^\alpha}\right). \tag{64}$$

Integrating the equilibrium equation (28) from r_1 to r_y and observing the boundary condition (36) at r_1, we obtain

$$p_1 - p_y = \lambda \int_{r_1}^{r_y} \frac{\sigma_e}{r}\,\mathrm{d}r, \tag{65}$$

where the notation $\sigma_r = -p_y$ for $r = r_y$ has been used. Differentiation with respect to time yields

$$\dot{p}_1 - \dot{p}_y = \lambda \int_{r_1}^{r_y} \frac{\dot{\sigma}_e}{r}\,\mathrm{d}r + \lambda\sigma_y \frac{\dot{r}_y}{r_y}, \tag{66}$$

where we have used $\sigma_e = \sigma_y$ for $r = r_y$. Differentiation of (66) gives

$$\ddot{p}_1 - \ddot{p}_y = \lambda \int_{r_1}^{r_y} \frac{\ddot{\sigma}_e}{r}\,\mathrm{d}r + \lambda(\dot{\sigma}_e)_{r_y}\frac{\dot{r}_y}{r_y} + \lambda\sigma_y\frac{\ddot{r}_y}{r_y} - \lambda\sigma_y\left(\frac{\dot{r}_y}{r_y}\right)^2. \tag{67}$$

From (65)–(67), we obtain

$$\begin{aligned}\lambda \int_{r_1}^{r_y} &\frac{\ddot{\sigma}_e + A\dot{\sigma}_e + B\sigma_e - F}{r}\,\mathrm{d}r \\ &= \ddot{p}_1 - \ddot{p}_y - \lambda(\dot{\sigma}_e)_{r_y}\frac{\dot{r}_y}{r_y} - \lambda\sigma_y\frac{\ddot{r}_y}{r_y} + \lambda\sigma_y\left(\frac{\dot{r}_y}{r_y}\right)^2 \\ &\quad + A\left(\dot{p}_1 - \dot{p}_y - \lambda\sigma_y\frac{\dot{r}_y}{r_y}\right) + B(p_1 - p_y) - \frac{\lambda}{\alpha}F\ln\left(\frac{r_y}{r_1}\right)^\alpha.\end{aligned} \tag{68}$$

By completely similar calculations, we obtain

$$\begin{aligned}\lambda \int_{r_y}^{r_2} &\frac{\ddot{\sigma}_e + A_e\dot{\sigma}_e + B_e\sigma_e}{r}\,\mathrm{d}r \\ &= \ddot{p}_y - \ddot{p}_2 + \lambda(\dot{\sigma}_e)_{r_y}\frac{\dot{r}_y}{r_y} + \lambda\sigma_y\frac{\ddot{r}_y}{r_y} - \lambda\sigma_y\left(\frac{\dot{r}_y}{r_y}\right)^2 \\ &\quad + A_e\left(\dot{p}_y - \dot{p}_2 + \lambda\sigma_y\frac{\dot{r}_y}{r_y}\right) + B_e(p_y - p_2).\end{aligned} \tag{69}$$

Inserting (68) and (69) in (64) gives

$$\begin{aligned}&\frac{\lambda}{\alpha}\left(\frac{1}{r_1^\alpha} - \frac{1}{r_2^\alpha}\right)(C\ddot{f} + D\dot{f}) + \ddot{p}_2 - \ddot{p}_1 + A_e\dot{p}_2 - A\dot{p}_1 + B_e p_2 - Bp_1 \\ &\quad + (A - A_e)\left[\dot{p}_y + \frac{\lambda}{\alpha}\sigma_y\left(\frac{r_y}{r_1}\right)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{r_y}{r_1}\right)^\alpha\right] + (B - B_e)p_y \\ &\quad + \frac{\lambda}{\alpha}F\ln\left(\frac{r_y}{r_1}\right)^\alpha = 0.\end{aligned} \tag{70}$$

It turns out to be possible to write the expressions $C\ddot{f} + D\dot{f}$ and p_y in terms of the plastic radius r_y. This transforms (70) into a differential equation from which r_y can be determined, but to perform this transformation we have to distinguish whether the plastic radius increases or decreases. Before proceeding with this, it becomes convenient to establish some relations for the initial conditions.

In the viscoelastic zone, we have initially an equation similar to (46), i.e.

$$\dot{\sigma}_{e0} = C\frac{\dot{f}_0}{r^\alpha} - (A_eC - D)\frac{f_0}{r^\alpha}. \tag{71}$$

Integration of the equilibrium equation (28) from r_1 and r_2 and subsequent differentiation yield at $t = 0^+$

$$\dot{p}_{10} - \dot{p}_{20} = \lambda\int_{r_1}^{r_{y0}} \frac{\dot{\sigma}_{e0}}{r}\,dr + \lambda\int_{r_{y0}}^{r_2} \frac{\dot{\sigma}_{e0}}{r}\,dr. \tag{72}$$

Use of (46) in the viscoplastic zone and (71) in the viscoelastic zone results in

$$\dot{p}_{10} - \dot{p}_{20} = \frac{\lambda}{\alpha}C\dot{f}_0\left(\frac{1}{r_1^\alpha} - \frac{1}{r_2^\alpha}\right) + \frac{\lambda}{\alpha}f_0\left[\frac{C(A - A_e)}{r_{y0}^\alpha} + \frac{A_eC - D}{r_2^\alpha} - \frac{AC - D}{r_1^\alpha}\right]$$
$$+ \frac{\lambda}{\alpha}N \ \ln\left(\frac{r_{y0}}{r_1}\right)^\alpha. \tag{73}$$

The plastic radius never decreases (pressurised vessels)

Usually, the loading of a pressurised vessel is such that the plastic zone will never decrease. This is the case, e.g. when the outer pressures are held constant. For such situations, the size of the viscoelastic zone will shrink or remain constant. That is, all material points in the viscoelastic region have always been in a viscoelastic state. Therefore, (60) applies in all the viscoelastic region and we can eliminate the unknown expression $Cf(t) + \psi(t)_e$ by observing that for $r = r_y$ (60) must give $\sigma_e = \sigma_y$. This implies that in all the viscoelastic zone we have

$$\sigma_e = \sigma_y\left(\frac{r_y}{r}\right)^\alpha. \tag{74}$$

Integrating the equilibrium equation (28) from r to r_2 and using the boundary condition (34) at r_2, we obtain in the viscoelastic zone

$$\sigma_r = -p_2 - \frac{\lambda}{\alpha}\sigma_y\left[\left(\frac{r_y}{r}\right)^\alpha - \left(\frac{r_y}{r_2}\right)^\alpha\right], \tag{75}$$

i.e. for $r = r_y$, we have

$$p_y = p_2 + \frac{\lambda}{\alpha}\sigma_y\left[1 - \left(\frac{r_y}{r_2}\right)^\alpha\right]. \tag{76}$$

Note that the stress distribution given by (74) and (75) corresponds exactly to that of a purely elastic vessel loaded by pressures p_y and p_2 at radius r_y and r_2, respectively.

By inserting (74) in constitutive equation (58), we obtain

$$C\ddot{f} + D\dot{f} = \sigma_y r_1^\alpha\left[\frac{d^2}{dt^2}\left(\frac{r_y}{r_1}\right)^\alpha + A_e\frac{d}{dt}\left(\frac{r_y}{r_1}\right)^\alpha + B_e\left(\frac{r_y}{r_1}\right)^\alpha\right]. \tag{77}$$

Use of (76) and (77) in (70) yields the expression sought for the development of the plastic radius

$$\left[1 - \left(\frac{r_1}{r_2}\right)^\alpha\right] \frac{d^2}{dt^2} \left(\frac{r_y}{r_1}\right)^\alpha + \left[A_e - A\left(\frac{r_1}{r_2}\right)^\alpha + (A - A_e)\left(\frac{r_y}{r_1}\right)^{-\alpha}\right] \frac{d}{dt}\left(\frac{r_y}{r_1}\right)^\alpha$$
$$+ \left[B_e - B\left(\frac{r_1}{r_2}\right)^\alpha\right]\left(\frac{r_y}{r_1}\right)^\alpha + \frac{F}{\sigma_y}\left[1 + \ln\left(\frac{r_y}{r_1}\right)^\alpha\right]$$
$$+ \frac{\alpha(\ddot{p}_2 - \ddot{p}_1)}{\lambda\ \sigma_y} + \frac{\alpha A(\dot{p}_2 - \dot{p}_1)}{\lambda\ \sigma_y} + \frac{\alpha B(p_2 - p_1)}{\lambda\ \sigma_y} = 0, \quad (78)$$

where $(B - B_e)\sigma_y = F$ has been used. This ordinary nonlinear differential equation of second order determines the development with time of the plastic radius so long as this radius never decreases. As shown later, this situation will be fulfilled, e.g. when the outer pressures p_1 and p_2 are constant with time. The solution of (78) requires two initial conditions to be derived in the following.

Initially, the linear elastic solution is given by (74) for $t = 0^+$, i.e.

$$\sigma_{e0} = \sigma_y\left(\frac{r_{y0}}{r}\right)^\alpha. \quad (79)$$

Integrating the equilibrium equation (28) at $t = 0^+$ from r_1 to r_2 and using the corresponding boundary conditions yields the first initial condition:

$$\left(\frac{r_{y0}}{r_1}\right)^\alpha = \frac{\alpha(p_{10} - p_{20})}{\lambda\sigma_y\left[1 - \left(\frac{r_1}{r_2}\right)^\alpha\right]}. \quad (80)$$

In the viscoelastic zone, (71) and (74) apply; i.e. we obtain

$$\dot{f}_0 = \frac{\sigma_y}{C} r_1^\alpha \left[\frac{d}{dt}\left(\frac{r_y}{r_1}\right)^\alpha\right]_0 + \left(A_e - \frac{D}{C}\right) f_0. \quad (81)$$

From (45) and (79), we have

$$f_0 = \frac{\sigma_y}{C} r_{y0}^\alpha. \quad (82)$$

Inserting (81) and (82) in (73) and using that $A - A_e = N/\sigma_y$, we finally obtain the second initial condition:

$$\left[\frac{d}{dt}\left(\frac{r_y}{r_1}\right)^\alpha\right]_0 = \frac{1}{1 - (r_1/r_2)^\alpha}\left\{\frac{\alpha(\dot{p}_{10} - \dot{p}_{20})}{\lambda\sigma_y} + \frac{N}{\sigma_y}\left[\left(\frac{r_{y0}}{r_1}\right)^\alpha - 1 - \ln\left(\frac{r_{y0}}{r_1}\right)^\alpha\right]\right\}. \quad (83)$$

Equations (80) and (83) provide the initial conditions for the ordinary differential equation of second order given by (78), which can be solved by simple, standard numerical means.

The coefficient to N/σ_y in (83) is easily shown to have a value between zero and unity. As expected, therefore, (83) shows that for constant boundary pressures, the plastic radius increases initially.

Assume that after a certain time, p_1 and p_2 become constant with time, i.e. $\ddot{p}_2 = \ddot{p}_1 = \dot{p}_2 = \dot{p}_1 = 0$. Assume, then, that the terms in the differential equation (78) that do not involve time rates become zero, i.e.

$$\left[B_e - B\left(\frac{r_1}{r_2}\right)^\alpha\right]\left(\frac{r_y}{r_1}\right)^\alpha + \frac{F}{\sigma_y}\left[1 + \ln\left(\frac{r_y}{r_1}\right)^\alpha\right] + \frac{\alpha B(p_2 - p_1)}{\lambda\ \sigma_y} = 0. \quad (84)$$

Dividing by B and using $F/(B\sigma_y) = 1/(1 + \eta_B/\eta_M)$ and $B_e/B = 1 - 1/(1 + \eta_B/\eta_M)$, we obtain

$$\left[1 - \left(\frac{r_1}{r_2}\right)^\alpha\right]\left(\frac{r_y}{r_1}\right)^\alpha + \frac{1}{1 + \eta_B/\eta_M}\left[1 + \ln\left(\frac{r_y}{r_1}\right)^\alpha - \left(\frac{r_y}{r_1}\right)^\alpha\right] + \frac{\alpha(p_2 - p_1)}{\lambda\,\sigma_y} = 0. \tag{85}$$

It is easily shown that the left-hand side of (85) is less than zero for $t = 0^+$ and increases with increasing $(r_y/r_1)^\alpha$ values. Therefore, as we consider only situations where the plastic radius does not decrease, we will always, when p_1 and p_2 become constant with time after a certain period, reach a situation where (85) is fulfilled. Consequently, the solution to differential equation (78) eventually will yield a solution where $\mathrm{d}/\mathrm{d}t(r_y/r_1)^\alpha = 0$. That is, we reach a state having a stationary plastic radius determined by (85). It is of interest to observe that this stationary plastic radius is determined by the ratios r_1/r_2, η_B/η_M and $(p_2 - p_1)/\sigma_y$ only. The last term is an expression for the normalised loading of the vessel, and the ratio of the Bingham to the Maxwell viscosity is the only material parameter of importance. That is, the Kelvin parameters have no influence on the stationary plastic radius.

By letting $\eta_B \to 0$ or $\eta_M \to \infty$, the ratio η_B/η_M becomes zero, and (85) takes the form

$$-\left(\frac{r_1}{r_2}\right)^\alpha\left(\frac{r_y}{r_1}\right)^\alpha + 1 + \ln\left(\frac{r_y}{r_1}\right)^\alpha + \frac{\alpha(p_2 - p_1)}{\lambda\,\sigma_y} = 0. \tag{86}$$

It is of interest to observe that this expression for the stationary plastic radius holds both for viscoelastic-ideal plastic behaviour ($\eta_M > 0$, $\eta_B = 0$) for elastic-viscoplastic behaviour ($\eta_M \to \infty$, $\eta_B > 0$) and for elastic-ideal plastic behaviour ($\eta_M \to \infty$, $\eta_B = 0$). For the latter case, the classical solution as given, for instance, by Hill[5] is rediscovered by (86). That is, for all the material behaviours just mentioned, the stationary plastic radius is equal to the classical elastic-ideal plastic solution.

After having discussed the determination of the plastic radius, we shall now determine the function $f(t)$ present, for instance, in the strain expressions (37) and (38). This function can be determined by means of (63), where the expression for the effective stress in the viscoelastic zone, i.e. (74), is inserted. This yields

$$f(t) = \frac{\sigma_y}{3\lambda}\left[\frac{3\lambda}{C}r_y^\alpha + \frac{e^{(-G_K/\eta_K)t}}{\eta_K}\int_0^t r_y^\alpha\, e^{(G_K/\eta_K)t}\,\mathrm{d}t + \frac{1}{\eta_M}\int_0^t r_y^\alpha\,\mathrm{d}t\right]. \tag{87}$$

Once the plastic radius has been determined using (78), a simple quadrature formula suffices to determine $f(t)$. Having determined $f(t)$, the function $\psi(t)$ is determined by means of (51). Use of (49) then provides the value of the effective stress in the viscoplastic zone directly. It should be noted, however, that (49) applies only in the region that has always been in a viscoplastic state; i.e. (49) applies from r_1 to the initial value of the plastic radius r_{y0} given by (80). In the viscoelastic zone from r_y to r_2, the effective stress is given by (74). Between r_{y0} and r_y, expressions for the effective stress can be found, but they turn out to be very complicated and for practical purposes are of minor importance only.

Integration of equilibrium equation (28) from r_1 to r using (49) and the boundary condition at r_1 gives

$$\sigma_r = -p_1 + \frac{\lambda}{\alpha}\phi(t)\ln\left(\frac{r}{r_1}\right)^\alpha + \frac{\lambda}{\alpha}\frac{Cf(t) + \psi(t)}{r_1^\alpha}\left[1 - \left(\frac{r_1}{r}\right)^\alpha\right] \tag{88}$$

valid between r_1 and r_{y0}. The functions $f(t)$ and $\psi(t)$ are evaluated as described above. In the viscoelastic region from r_y to r_2, the radial stress is given by (75).

The stress distribution has now been determined completely. As $f(t)$ is known through (87), it follows from (31), (37) and (38) that the strain field is also known completely.

If the outer pressures become constant with time, the plastic radius will eventually, as shown previously, reach a stationary value given by (85). This in turn implies that the stress field in the viscoelastic zone becomes stationary [see (74) and (75)]. For the stationary state, we therefore have $\dot{p}_1 = \dot{p}_y = \dot{r}_y = 0$. Equation (66) then shows that the effective stress in the viscoplastic zone also must be stationary. It is of importance that very simple closed-form solutions can be derived for the stress field in the total viscoplastic zone from r_1 to r_y, when a stationary state has been reached. These expressions can be obtained as follows.

Constitutive equation (39) applies to all the viscoplastic region. When the stationary state has been reached, we have $\ddot{\sigma}_e = \dot{\sigma}_e = 0$, and (39) becomes

$$B\sigma_e = \frac{C\ddot{f} + D\dot{f}}{r^\alpha} + F. \tag{89}$$

In the stationary state, $\ddot{r}_y = \dot{r}_y = 0$, and use of (77) yields

$$\sigma_e = \frac{\sigma_y}{1 + \eta_B/\eta_M}\left[1 - \left(\frac{r_y}{r}\right)^\alpha\right] + \sigma_y\left(\frac{r_y}{r}\right)^\alpha, \tag{90}$$

where the relations $B_e/B = 1 - 1/(1 + \eta_B/\eta_M)$ and $F/B = \sigma_y/(1 + \eta_B/\eta_M)$ have been applied. This stationary stress state is valid in the total viscoplastic region from r_1 to r_y. The above derivation is based on the assumption that $B \neq 0$. B becomes equal to 0 only when $\eta_K \to \infty$, i.e. when the Kelvin response is suppressed. In that case, differential equation (39) degenerates to a first-order differential equation, and it is easily shown that (90) also provides the stationary value in that case. The stationary effective stress in the viscoelastic region from r_y to r_2 is still given by (74).

Integration of equilibrium equation (28) using (90) and the boundary condition at r_1 yields the stationary radial stress field:

$$\sigma_r = -p_1 + \frac{\lambda\,\sigma_y}{\alpha(1 + \eta_B/\eta_M)}\left\{\frac{\eta_B}{\eta_M}\left[\left(\frac{r_y}{r_1}\right)^\alpha - \left(\frac{r_y}{r}\right)^\alpha\right] + \ln\left(\frac{r}{r_1}\right)^\alpha\right\} \tag{91}$$

valid in the total viscoplastic region from r_1 to r_y. The stationary radial stress in the viscoelastic zone from r_y to r_2 is still given by (75).

It should be observed that the properties of the Kelvin element do not enter the expressions for the stationary stresses and plastic radius; this is so because the Kelvin element for constant stresses acts eventually as a rigid element. The only material parameters that influence the stationary stress state are the η_B/η_M ratio and the yield stress σ_y.

An illustration of the stationary effective stress distribution is provided by Fig. 2 for a sphere. Note that inclusion of the Maxwell element enables one to simulate widely different stationary stress fields, ranging from the purely elastic to the elastic-ideal plastic ones.

The solution to the spherical problem is given here for any Poisson's ratio. It may be observed, however, that neither the initial elastic nor the stationary stress state depends on Poisson's ratio. This suggests Poisson's ratio to be of minor importance for transient stress states also.

That the elastic parameters influence the transient stress field is a result of the stress redistribution that takes place. This redistribution depends on the relative stiffnesses of the elastic and viscous components and therefore also on the elastic parameters.

In the stationary situation, a simple expression can be given for the stationary

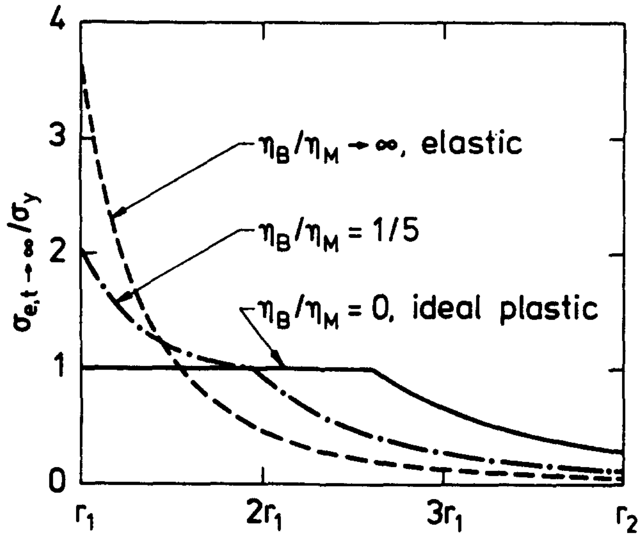

Fig. 2. Stationary effective stress field for a pressurised sphere: $r_1/r_2 = 1/4$, $(p_1 - p_2)/\sigma_y = 12/5$.

strain rate also. For stationary stresses, (37) shows that $\dot{\epsilon}_\theta = \dot{f}/r^\alpha$. As the response of the Kelvin element is completely rigid, the stationary strain rate can be derived as if the Kelvin element did not exist. Consequently, differentiation of (87) yields

$$\dot{\epsilon}_{\theta,t\to\infty} = \frac{\sigma_y}{3\lambda\eta_M}\left(\frac{r_y}{r}\right)^\alpha, \tag{92}$$

where r_y is the stationary plastic radius determined by (85). The stationary state is therefore described by constant stresses and constant strain rates. The vital importance of the Maxwell viscosity, η_M, in (92) can be explained by the restraining effect of the viscoelastic zone surrounding the viscoplastic zone. It appears that if the Maxwell viscosity is suppressed, i.e. $\eta_M \to \infty$, the strains approach a stationary state for large times.

It can also be observed that the parameter values of the Kelvin element influence neither the stationary stress field nor the stationary strain rate field. On the other hand, the magnitude of the Maxwell viscosity is of major importance for the long-term behaviour.

We shall now compare the results obtained herein with those previously given in the literature for simplified material behaviours. Consider first suppression of viscoplasticity that reduces the results to those valid for a Burgers material. When $\eta_B \to \infty$, we have $N \to 0$; i.e. (50) becomes

$$\phi(t)_{\eta_B\to\infty} = 0. \tag{93}$$

Moreover, as $\eta_B \to \infty$ implies that $R' = R'_e$ and $R'' = R''_e$, it appears from (51) and (61) that $\psi(t) = \psi(t)_e$. A comparison of (49) with (60) then shows, as expected, that the stress field in the viscoelastic and viscoplastic zone becomes identical. This stress field is therefore given by (74). We then rediscover that for a Burgers material, the stress field is similar to the elastic distribution. To determine the development of the plastic radius, we observe that $\eta_B \to \infty$ indicates that $A = A_e$, $B = B_e$ and $F = 0$; i.e. (78) directly provides the solution

$$\left(\frac{r_y}{r_1}\right)^\alpha_{\eta_B\to\infty} = \frac{\alpha(p_1 - p_2)}{\lambda\sigma_y\,[1 - (r_1/r_2)^\alpha]}, \tag{94}$$

which satisfies the initial conditions (80) and (83). That is, the stress field becomes equal to the elastic field.

To evaluate the strain field (37) and (38) for a Burgers material, the function $f(t)$ is obtained by means of (87) using (94). This yields

$$f(t)_{\eta_B\to\infty} = \frac{\alpha}{3\lambda^2\,[1-(r_1/r_2)^\alpha]}\,r_1^\alpha\left[\frac{3\lambda}{C}(p_1-p_2) + \frac{e^{(-G_K/\eta_K)t}}{\eta_K}\int_0^t (p_1-p_2)\,e^{(G_K/\eta_K)t}\,dt + \frac{1}{\eta_M}\int_0^t (p_1-p_2)\,dt\right], \quad (95)$$

and for constant pressure loadings, we obtain

$$f(t)_{\eta_B\to\infty} = \frac{\alpha(p_1-p_2)}{3\lambda^2\,[1-(r_1/r_2)^\alpha]}\,r_1^\alpha\left\{\frac{3\lambda}{C} + \frac{1}{G_K}\,[1-e^{(-G_K/\eta_K)t}] + \frac{t}{\eta_M}\right\}. \quad (96)$$

For the cylindrical problem, this solution is identical to that given by Gnirk and Johnson[6].

Consider, then, the situation in which the viscoplastic behaviour degenerates into the ideal plastic behaviour; i.e. let $\eta_B \to 0$. This implies that $N \to \infty$ [see (47)]. This in turn suggests that the initial rate of the plastic radius becomes infinite [see (83)]; i.e. the stationary value of the plastic radius given by (86) applies immediately. It is of interest that (86) holds for all times both for elastic-ideal plastic behaviour ($\eta_M \to \infty$, $\eta_B = 0$) and for viscoelastic-ideal plastic behaviour ($\eta_M > 0$, $\eta_B = 0$). This conclusion is in agreement with the finding of Crochet[7]. For constant loading, the plastic radius is also constant; i.e. (87) results in

$$f(t)_{\eta_B\to 0} = \frac{\sigma_y}{3\lambda}\,r_y^\alpha\left\{\frac{3\lambda}{C} + \frac{1}{G_K}\,[1-e^{(-G_K/\eta_K)t}] + \frac{t}{\eta_M}\right\} \quad (97)$$

valid for a viscoelastic-ideal plastic material. A comparison with (96) shows similarities as only the coefficient to the bracket differs from that of a Burgers material. The coefficient in (97) is larger than that of a Burgers material owing to the larger flexibility.

Let us now consider the more intriguing cases where we have the problem of a moving plastic boundary and different simplified material behaviours.

If the Kelvin element is suppressed, the general differential equation (78) can be simplified considerably. Having $\eta_K \to \infty$ implies that $B = B_e = F = 0$ as well as $A - A_e = N/\sigma_y$, and this enables one to make an integration of (78). Using the initial condition given by (83), this integration yields, after some algebra,

$$\left[1-\left(\frac{r_1}{r_2}\right)^\alpha\right]\frac{d}{dt}\left(\frac{r_y}{r_1}\right)^\alpha + A\left[1-\left(\frac{r_1}{r_2}\right)^\alpha\right]\left(\frac{r_y}{r_1}\right)^\alpha + \frac{N}{\sigma_y}\left[1+\ln\left(\frac{r_y}{r_1}\right)^\alpha - \left(\frac{r_y}{r_1}\right)^\alpha\right] + \frac{\alpha(\dot p_2-\dot p_1)}{\lambda\,\sigma_y} + \frac{\alpha A(p_2-p_1)}{\lambda\,\sigma_y} = 0. \quad (98)$$

This first-order differential equation has to be integrated using the initial condition given by (80).

A further simplification is obtained by also suppressing the Maxwell viscosity. Letting $\eta_M \to \infty$, we have $A_{\eta_K\to\infty,\eta_M\to\infty} = N/\sigma_y$; i.e. (98) reduces to

$$\left[1-\left(\frac{r_1}{r_2}\right)^\alpha\right]\frac{d}{dt}\left(\frac{r_y}{r_1}\right)^\alpha + \frac{N}{\sigma_y}\left[1-\left(\frac{r_1}{r_2}\right)^\alpha + \ln\left(\frac{r_y}{r_1}\right)^\alpha\right] + \frac{\alpha(\dot p_2-\dot p_1)}{\lambda\,\sigma_y} + \frac{\alpha N(p_2-p_1)}{\lambda\,\sigma_y^2} = 0 \quad (99)$$

valid for an elastic-viscoplastic material. The initial condition is still given by (80).

Solutions have been given previously in the literature for this simplified elastic-viscoplastic behaviour. Madejski[2] treated a sphere, but the results of this often-referred-to paper do not correspond to that of (99). However, as previously discussed, Madejski did not observe that the stress state in the viscoplastic region is dependent on whether the stress state always has been in that state or whether it previously was in an elastic state. The results in [2] are therefore erroneous. In addition, Berest and Nguyen[3] very recently also treated the elastic-viscoplastic behaviour of an infinite sphere and their results are in agreement with (99).

A rather comprehensive exposition of the behaviour of pressurised viscoelastic-viscoplastic vessels having a viscoelastic zone has been given above. The outer pressures are assumed to be constant or they might change with time, but this variation must be such that the plastic radius never decreases.

In the following section, we shall treat the case of viscoelastic-viscoplastic vessels, where the plastic radius never increases.

The plastic radius never increases (relaxation loaded vessels)

For certain loadings, the plastic zone will never increase. This occurs, for instance, when the displacement of the inner boundary of the vessel is kept constant with time, i.e. when the vessel is subjected to relaxation conditions. For such cases, all material points in the viscoplastic region have always been in a viscoplastic state. Therefore, (49) applies in all the viscoplastic region, and we can eliminate the unknown term $Cf(t) + \psi(t)$ by observing that for $r = r_y$, (49) must give $\sigma_e = \sigma_y$. This implies that in the viscoplastic zone we have

$$\sigma_e = \phi(t)\left[1 - \left(\frac{r_y}{r}\right)^{\alpha}\right] + \sigma_y\left(\frac{r_y}{r}\right)^{\alpha}, \tag{100}$$

where the time-dependent function $\phi(t)$ is given by (50). Integrating equilibrium equation (28) from r_1 to r and using boundary condition (36) at r_1, we obtain in the viscoplastic zone

$$\sigma_r = -p_1 + \frac{\lambda}{\alpha}\phi(t)\ln\left(\frac{r}{r_1}\right)^{\alpha} + \frac{\lambda}{\alpha}\left[\left(\frac{r_y}{r}\right)^{\alpha} - \left(\frac{r_y}{r_1}\right)^{\alpha}\right][\phi(t) - \sigma_y]. \tag{101}$$

That is, for $r = r_y$, we have

$$p_y = p_1 - \frac{\lambda}{\alpha}\phi(t)\ln\left(\frac{r_y}{r_1}\right)^{\alpha} - \frac{\lambda}{\alpha}\left[1 - \left(\frac{r_y}{r_1}\right)^{\alpha}\right][\phi(t) - \sigma_y]. \tag{102}$$

If pressurised vessels were considered, we could proceed just as in the previous section; i.e. (100) and (102) would replace (74) and (76). Instead of (78), we would obtain another differential equation, which would determine the development of the plastic radius. However, as we consider here loadings in which the plastic radius never increases, this would correspond to loading conditions where the outer pressure difference decreased rapidly with time. Such pressure conditions are seldom encountered in practice, and therefore we shall instead treat a more interesting loading case, namely, when the vessel is subjected to relaxation conditions. This case fulfills the prerequisite that the plastic zone never increase.

To treat the relaxation condition, it becomes necessary to assume incompressibility for the spherical problem also. It is possible to evaluate the response even when the prescribed displacement at the inner boundary varies with time, but a particularly simple solution follows if the prescribed displacement is kept constant with time; i.e. the boundary condition at the inner radius is u_1 = constant. This condition is most often met in applications.

From the expression of the circumferential strain (37) and using the condition of incompressibility, i.e. $M = 0$, we obtain

$$f = \frac{u_1}{r_1} r_1^\alpha, \tag{103}$$

i.e. f takes a constant value. The effective stress in the viscoplastic zone has already been determined by (100). However, we can obtain another expression by using (49) and evaluating the function $\psi(t)$ by means of (51) using the constant value of f given by (103). This yields

$$\sigma_e = \phi(t) - \frac{u_1}{r_1}\left(\frac{r_1}{r}\right)^\alpha \frac{1}{R' - R''}\left[(CR'' + D)\,e^{R''t} - (CR' + D)\,e^{R't}\right]. \tag{104}$$

The two expressions for the effective stress, (100) and (104), must be identical; this implies

$$\left(\frac{r_y}{r_1}\right)^\alpha = -\frac{u_1}{r_1}\frac{1}{(R' - R'')\,[\sigma_y - \phi(t)]}\left[(CR'' + D)\,e^{R''t} - (CR' + D)\,e^{R't}\right]. \tag{105}$$

This simple closed-form expression determines the development with time of the plastic radius. As expected, it can be shown that the plastic radius is a never-increasing function of time.

The initial value of the plastic radius follows from (105) with $t = 0^+$, i.e.

$$\left(\frac{r_{y0}}{r_1}\right)^\alpha = \frac{u_1\,C}{r_1\,\sigma_y} \tag{106}$$

in agreement with the linear elastic solution.

The outer pressure $p_2(t)$ is an arbitrary known function of time. However, the variation of the so-called shrink-fit pressure, i.e. the inner pressure $p_1(t)$, is still unknown. This pressure shall now be determined.

Inserting the expression for p_y given by (102) in (70) and using $\ddot{f} = \dot{f} = 0$, we obtain

$$\begin{aligned}
&\frac{\alpha}{\lambda}(\ddot{p}_2 - \ddot{p}_1) + \frac{\alpha}{\lambda}A_e(\dot{p}_2 - \dot{p}_1) + \frac{\alpha}{\lambda}B_e(p_2 - p_1)\\
&+ (A - A_e)(\sigma_y - \phi)\left[\left(\frac{r_y}{r_1}\right)^{-\alpha} - 1\right]\frac{d}{dt}\left(\frac{r_y}{r_1}\right)^\alpha\\
&+ [(A - A_e)\dot{\phi} - (B - B_e)(\sigma_y - \phi)]\left(\frac{r_y}{r_1}\right)^\alpha\\
&+ [F - (A - A_e)\dot{\phi} - (B - B_e)\phi]\ln\left(\frac{r_y}{r_1}\right)^\alpha\\
&+ (B - B_e)(\sigma_y - \phi) - (A - A_e)\dot{\phi} = 0.
\end{aligned} \tag{107}$$

This ordinary linear differential equation of second order determines the pressure difference $p_2 - p_1$ and thereby the variation of the shrink-fit pressure $p_1(t)$. The plastic radius r_y and the function $\phi(t)$ are given by the explicit expressions (105) and (50), respectively. The solution of (107) requires two initial conditions to be derived in the following.

The initial value of the plastic radius is determined by the linear elastic response given by (80), i.e.

$$p_{10} - p_{20} = \frac{\lambda}{\alpha}\sigma_y\left[1 - \left(\frac{r_1}{r_2}\right)^\alpha\right]\left(\frac{r_{y0}}{r_1}\right)^\alpha, \tag{108}$$

where the term $(r_{y0}/r_1)^\alpha$ is given by (106). The second initial condition is obtained by means of (73), using f as a constant given by (103), i.e.

$$\dot{p}_{10} - \dot{p}_{20} = \frac{\lambda}{\alpha}\sigma_y\left(\frac{r_{y0}}{r_1}\right)^\alpha\left[(A - A_e)\left(\frac{r_{y0}}{r_1}\right)^{-\alpha} + \left(A_e - \frac{D}{C}\right)\left(\frac{r_1}{r_2}\right)^\alpha - A + \frac{D}{C}\right] + \frac{\lambda}{\alpha}N\ln\left(\frac{r_{y0}}{r_1}\right)^\alpha . \quad (109)$$

Equations (108) and (109) provide the initial conditions for the ordinary linear differential equation (107), which can be solved by simple numerical means. Consequently, (105) determines the development of the plastic radius, and (107) determines the variation of the shrink-fit pressure $p_1(t)$.

The strain field is obtained directly from (37) and (38) using (103) and $M = 0$. The stress field in all the viscoplastic zone is given by (100) and (101). The viscoelastic zone consists of two fundamental different regions. One region is between the current plastic radius and the initial plastic radius, i.e. $r_y \leq r \leq r_{y0}$, where the stress state initially was in a viscoplastic state and now is in a viscoelastic state. Because of this change in the material behaviour, the stress state is very complicated to determine. In the other region, $r_{y0} \leq r \leq r_2$, the stress state has always been in a viscoelastic state; i.e. we can determine the stress state directly by means of solution (60), where $\psi(t)_e$ is given by (61). As $f(t)$ in the present situation is a constant given by (103), we can evaluate $\psi(t)_e$ and obtain

$$\sigma_e = -\frac{u_1}{r_1}\left(\frac{r_1}{r}\right)^\alpha \frac{1}{R'_e - R''_e}\left[(CR''_e + D)\,e^{R'_e t} - (CR'_e + D)\,e^{R''_e t}\right] \quad (110)$$

valid in the region $r_{y0} \leq r \leq r_2$. Integrating the equilibrium equation (28) from r to r_2 and using the boundary condition (34) at r_2, we obtain

$$\sigma_r = -p_2 - \frac{\lambda}{\alpha}\frac{u_1}{r_1}\left[\left(\frac{r_1}{r_2}\right)^\alpha - \left(\frac{r_1}{r}\right)^\alpha\right]\frac{1}{R'_e - R''_e} \times \left[(CR''_e + D)\,e^{R'_e t} - (CR'_e + D)\,e^{R''_e t}\right] \quad (111)$$

valid in the region $r_{y0} \leq r \leq r_2$. This completes the determination of the stress and strain fields in relaxation loaded vessels.

It is of interest that (110) and (111) demonstrate that the stress field in the viscoelastic region $r_{y0} \leq r \leq r_2$ depends only on the viscoelastic parameters; i.e. this stress field is independent of the plastic radius and the viscoplastic response. This is a consequence of the assumptions of incompressibility and a constant prescribed displacement at the inner boundary, implying that the strain field is constant with time.

Let us now evaluate the plastic radius for large times. Equation (50) shows that

$$\phi(t)_{t\to\infty} = \frac{\sigma_y}{1 + \eta_B/\eta_M}, \quad (112)$$

and (105) then implies that for $\eta_B/\eta_M > 0$ the plastic radius will tend to approach zero; i.e. the plastic zone will eventually disappear. The following relaxation behaviour is then determined by a viscoelastic response.

This result suggests that the Maxwell viscosity η_M plays a vital role in the relaxation behaviour. By suppressing the Maxwell viscosity, i.e. $\eta_M \to \infty$, the expression for $\phi(t)$, (50), infers that (105) can be written as

$$\left(\frac{r_y}{r_1}\right)^\alpha_{\eta_M\to\infty} = -\frac{u_1}{r_1} \times \frac{(CR'' + D)\,e^{(R''-R')t} - (CR' + D)}{(\sigma_y R' + N)\,e^{(R''-R')t} - (\sigma_y R'' + N)} \quad (113)$$

i.e. for large times we obtain

$$\left(\frac{r_y}{r_1}\right)^{\alpha}_{\eta_M\to\infty, t\to\infty} = \left(\frac{r_{y0}}{r_1}\right)^{\alpha} \times \frac{R'' + G_M(1/\eta_K + 1/\eta_B)}{R'' + G_M/\eta_B}, \tag{114}$$

where the relations $R' = -R'' - A$, $D/C = G_K/\eta_K$ and (106) have been used and where the expressions for A, (40), and N, (47), have been applied, observing that $M = 0$ holds. Thus, when the Maxwell viscosity is suppressed, we might reach a situation with a stationary plastic radius $r_y \geq r_1$, and as (100) and (112) apply we also obtain a stationary effective stress distribution $\sigma_e = \sigma_y$ throughout the viscoplastic zone. When $\eta_M \to \infty$, then $B_e = 0$, i.e. $R'_e = 0$. Consequently, (110) indicates that also in the viscoelastic zone we obtain a stationary stress distribution; i.e. the pressure difference $p_2 - p_1$ also approaches a stationary value. For pressurised vessels, the Maxwell viscosity was previously found to be of vital importance for the stationary state of the stress and strain rate [see the discussion of (90) and (92)]. The discussion above demonstrates the importance of the Maxwell viscosity for relaxation loaded vessels as well. If the Maxwell viscosity is suppressed, a stationary relaxation state might be achieved. This observation has far-reaching consequences, e.g. for tunnel-lining problems, because Maxwell viscosity is often not considered in such problems (see [4, 8]).

Let us now consider the response for some simplified material behaviours and compare the results with those previously given in the literature.

If both the Maxwell and Kelvin viscosities are suppressed, we obtain an elastic-viscoplastic material and a particularly simple situation arises. By letting $\eta_K \to \infty$, (114) shows that the plastic radius is constant with time. This reduces the problem considerably because one avoids the delicate problem of having a moving plastic boundary. This simple elastic-viscoplastic response was treated by Nonaka[4] for infinite vessels. Here we can state the results for such a situation directly. When $\eta_M \to \infty$ and $\eta_K \to \infty$, we have $B = 0$, i.e. $R' = 0$ and $R'' = -A = -N/\sigma_y$. This implies that (50) reduces to

$$\phi(t)_{\eta_M\to\infty, \eta_K\to\infty} = \sigma_y\,[1 - e^{(-N/\sigma_y)t}] \tag{115}$$

to be used in the stress expressions (100) and (101) in the viscoplastic zone. In addition, we have $R'_e = R''_e = D = 0$, i.e. $e^{R'_e t} = e^{R''_e t} = 1$. This implies that outside the viscoplastic zone, where (110) and (111) apply, these expressions reduce to

$$\sigma_e = \sigma_y \left(\frac{r_y}{r}\right)^{\alpha} \tag{116}$$

$$\sigma_r = -p_2 - \frac{\lambda}{\alpha}\sigma_y \left[\left(\frac{r_y}{r}\right)^{\alpha} - \left(\frac{r_y}{r_2}\right)^{\alpha}\right], \tag{117}$$

where (106) has been used. Thus, we retrieve an elastic stress distribution, as expected.

To determine the pressure difference $p_2 - p_1$, it is possible to integrate (107) directly, using $A_e = B_e = B = F = 0$ and $A = N/\sigma_y$. However, it becomes more convenient to observe that (117) for $r = r_y$ must yield $\sigma_r = -p_y$, where p_y is given by (102). This results in

$$p_2 - p_1 = -\frac{\lambda}{\alpha}\sigma_y \left\{\ln\left(\frac{r_y}{r_1}\right)^{\alpha} - \left(\frac{r_y}{r_2}\right)^{\alpha} + 1 + \left[\left(\frac{r_y}{r_1}\right)^{\alpha} - 1 - \ln\left(\frac{r_y}{r_1}\right)^{\alpha}\right] e^{(-N/\sigma_y)t}\right\}. \tag{118}$$

This result is easily shown to satisfy differential equation (107) as well as its initial conditions (108) and (109). Observing that $M = 0$, i.e. $N/\sigma_y = G_M/\eta_B$, (118) confirms the results of Nonaka[4].

Let us now evaluate the relaxation behaviour when the viscoplasticity is suppressed, i.e. for a Burgers material. For $\eta_B \to \infty$, $\phi(t) = 0$ applies [see (93)], as well as $R' = R'_e$ and $R'' = R''_e$, i.e. a comparison of (104) with (110) confirms that we have the same stress distribution in the viscoelastic and viscoplastic zone. Equation (100) shows that the stress field becomes

$$\sigma_e = \sigma_y \left(\frac{r_y}{r}\right)^{\alpha}, \tag{119}$$

which, as expected, corresponds to a linear elastic distribution. The plastic radius is still determined by means of (105), observing that $\phi(t) = 0$ and $R' = R'_e$, $R'' = R''_e$.

When $\eta_B \to \infty$, we have $A = A_e$, $B = B_e$ and $F = 0$; i.e. (107) degenerates to

$$\ddot{p}_2 - \ddot{p}_1 + A_e(\dot{p}_2 - \dot{p}_1) + B_e(p_2 - p_1) = 0, \tag{120}$$

and it is easily shown that the solution to this differential equation

$$p_1 - p_2 = \frac{\lambda}{\alpha}\sigma_y\left[1 - \left(\frac{r_1}{r_2}\right)^{\alpha}\right]\left(\frac{r_y}{r_1}\right)^{\alpha} \tag{121}$$

satisfies initial conditions (108) and (109). The plastic radius is still determined by means of (105), observing that $\phi(t) = 0$ and $R' = R'_e$, $R'' = R''_e$. The solution above refers to relaxation loaded vessels made of a Burgers material. When $\eta_K \to \infty$, it is easily shown that the expressions reduce to those given by Davis[9] for the relaxation of cylinders made of a Maxwell material.

CONCLUSIONS

Solutions for the viscoelastic-viscoplastic behaviour of partly plastic thick-walled cylinders in plane strain and spheres were derived for the stress and strain fields. The loading was due either to pressurisation or relaxation.

For deviatoric loading, the constitutive model consists of a viscoelastic Burgers element in combination with a viscoplastic Bingham element. For volumetric loading, linear elastic behaviour was assumed for the spherical problem, whereas incompressibility was assumed for the cylindrical one. The Kelvin part of the Burgers element exhibits primary, reversible creep, whereas both the Maxwell part of the Burgers element as well as the Bingham element exhibit secondary, irreversible creep. Thus, the constitutive model reflects many creep characteristics observed in a variety of materials.

The loadings considered were quite general, as the outer pressures might vary with time. Alternatively, the prescribed inner displacement was kept constant. This, in connection with the rather general constitutive model, leads to a variety of applications. In addition, it has been shown that it is possible to follow a unified treatment for a fairly long time even for the variety of loading cases considered.

It turns out that the analysis of partly plastic vessels is impeded by the problem of having a moving plastic boundary. This suggests the analysis of fully plastic vessels to be much simpler, which is confirmed by the treatment given in [1]. On the other hand, [1] demonstrates that much of the analysis presented here can be applied directly to fully plastic vessels.

Apart from a detailed derivation of the stress and strain fields in partly plastic vessels, a major effort was given to a discussion of principal aspects of the vessel behaviour and of the influence of the different material parameters.

For the pressurised vessels, a stationary stress and strain rate situation arises for large times. The magnitude of the Maxwell viscosity turns out to be of major importance for these stationary states.

For relaxation loaded vessels, the Maxwell viscosity is also of decisive importance. If this viscosity is ignored, the relaxation behaviour may approach a stationary stress and strain state. If Maxwell viscosity is included, the vessels will always experience a continued relaxation.

The significance of the other material parameters was also discussed, and it was shown that for a hierarchy of simplified material behaviours, the derived solutions degenerate to a variety of previously obtained results.

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APPENDIX A

We shall give here the solution to constitutive equation (39) subjected to initial conditions (45) and (46). Lengthy but trivial calculations, using $R'R'' = B$ [see (48)], show that the complete solution to (39) can be written as

$$\sigma_e = c_1(r)\,e^{R't} + c_2(r)\,e^{R''t} + \frac{Cf(t) + \psi(t)}{r^\alpha} + \frac{F}{B}, \tag{A1}$$

where $c_1(r)$ and $c_2(r)$ are arbitrary functions depending on the radius, whereas the time-dependent function $\psi(t)$ is given by

$$\psi(t) = -\frac{1}{R' - R''}\left[R''(CR'' + D)\,e^{R''t}\int_0^t f(t)\,e^{-R''t}\,dt - R'(CR' + D)\,e^{R't}\int_0^t f(t)\,e^{-R't}\,dt\right], \tag{A2}$$

where $\psi_0 = 0$ and $\dot{\psi}_0 = f_0(D - AC)$ apply.

The c_1 and c_2 functions can be determined as functions of the initial conditions given by $\sigma_{e,0}$ and $\dot{\sigma}_{e,0}$. This results in

$$\sigma_e = \frac{e^{R't} - e^{R''t}}{R' - R''}\left[\dot{\sigma}_{e,0} - \left(\sigma_{e,0} - \frac{Cf_0}{r^\alpha} - \frac{F}{B}\right)R'' - \frac{C\dot{f}_0 + (D - AC)f_0}{r^\alpha}\right] + \left(\sigma_{e,0} - \frac{Cf_0}{r^\alpha} - \frac{F}{B}\right)e^{R''t} + \frac{Cf(t) + \psi(t)}{r^\alpha} + \frac{F}{B}. \tag{A3}$$

Finally, using the initial conditions as specified by (45) and (46), we obtain

$$\sigma_e = \frac{e^{R't} - e^{R''t}}{R' - R''}\left(N + \frac{F}{B}R''\right) + \frac{F}{B}(1 - e^{R''t}) + \frac{Cf(t) + \psi(t)}{r^\alpha}. \tag{A4}$$

APPENDIX B

In this appendix, we shall derive an expression for the time-dependent function $f(t)$. For this purpose, consider constitutive equation (39) valid for a viscoelastic-viscoplastic stress state:

$$\frac{\ddot{f}}{r^\alpha} + \frac{D}{C}\frac{\dot{f}}{r^\alpha} = \frac{\ddot{\sigma}_e}{C} + \frac{A}{C}\dot{\sigma}_e + \frac{B}{C}\sigma_e - \frac{F}{C}. \tag{B1}$$

Using

$$\int e^{(D/C)t}\left(\int \sigma_e \, dt\right) dt = \frac{C}{D} e^{(D/C)t} \int \sigma_e \, dt - \frac{C}{D}\int \sigma_e e^{(D/C)t} \, dt,$$

it is easily shown that the solution to differential equation (B1) is

$$\begin{aligned}\frac{f(t)}{r^\alpha} &= \frac{\sigma_e}{C} + e^{-(D/C)t}\left(\frac{A}{C} - \frac{D}{C^2} - \frac{B}{D}\right)\int_0^t \sigma_e e^{(D/C)t} \, dt + \frac{B}{D}\int_0^t \sigma_e \, dt \\ &\quad - \frac{F}{D}t + \frac{FC}{D^2} + K_1(r) + K_2(r) e^{-(D/C)t},\end{aligned} \tag{B2}$$

where $K_1(r)$ and $K_2(r)$ are arbitrary functions to be determined from the initial conditions. These conditions emerge from (45) and (46), and they imply that

$$K_1(r) = -\frac{N}{D}$$

$$K_2(r) = \frac{N}{D} - \frac{FC}{D^2}.$$

Inserting these expressions in (B2) and using (40)–(44) as well as (47), we derive

$$\begin{aligned}\frac{f(t)}{r^\alpha} = \frac{1}{3\lambda}\Bigg[&\frac{3\lambda}{C}\sigma_e + \frac{e^{(-G_K/\eta_K)t}}{\eta_K}\int_0^t \sigma_e e^{(G_K/\eta_K)t} \, dt \\ &+ \left(\frac{1}{\eta_M} + \frac{1}{\eta_B}\right)\int_0^t \sigma_e \, dt - \frac{\sigma_y}{\eta_B}t\Bigg].\end{aligned} \tag{B3}$$

The derivation above is based on the assumption that $D \neq 0$. However, $D = 0$ when $\eta_K \to \infty$, and it is easily shown that (B3) is also the correct solution to differential equation (B1) in this simplified situation.

Note that solution (B3) applies to material points, which initially are in a viscoelastic-viscoplastic state and which remain in such a state.